Recall:

A function $f: [a,b] \to \mathbb{R}$ is integrable

if

$$U(f) = L(f)$$

$$U(f) = \inf_{P \text{ partition of } [a,b]} U(f,P) \qquad L(f) = \sup_{P \text{ partition of } [a,b]} L(f,P)$$

Useful criterion:

<u>Theorem</u> f integrable $\Longleftrightarrow$

$\forall \varepsilon > 0$ $\exists$ partition P such that

$$U(f,P) - L(f,P) < \varepsilon$$

Def. $f : S \subset \mathbb{R} \to \mathbb{R}$ monotonic

if either $f(x) \le f(y)$ for all $x < y$ (mon. increasing)

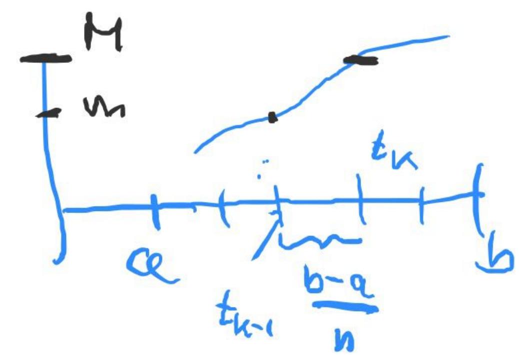
or $f(x) \ge f(y)$ " " " (" decreasing)

Theorem f monotonic $\Rightarrow$ f integrable

Proof. pick $\varepsilon > 0$ pick n such that $\boxed{\frac{b-a}{n} < \frac{\varepsilon}{f(b)-f(a)}}$

Define partition $P = \{t_0, t_1 \ldots t_n\}$ by

$$t_j = a + j\,\frac{b-a}{n}$$



do proof for f mon. increasing

$M(f, [t_{k-1}, t_k]) = \sup\{f(x), x \in [t_{k-1}, t_k]\} = f(t_k)$

$m(f, \ldots) = \inf\{\;"\;\;"\;\} = f(t_{k-1})$

$$\Rightarrow \quad M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) = f(t_k) - f(t_{k-1})$$

$$\Rightarrow U(f,P) - L(f,P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \underbrace{(t_k - t_{k-1})}_{= \frac{b-a}{n}}$$

$$= \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1}) \right) \cdot \frac{b-a}{n}$$

$$= \left[\cancel{f(t_1)} - f(t_0) + \cancel{f(t_2)} - \cancel{f(t_1)} + f(t_3) - \cancel{f(t_2)} \ldots f(t_{n-1}) - f(t_n) \right] \underbrace{}_{\frac{b-a}{n}}$$

$$= \left[\underset{b}{\overset{\shortparallel}{f(t_n)}} - \underset{a}{\overset{\shortparallel}{f(t_0)}} \right] \frac{b-a}{n}$$

$$< \left[f(b) - f(a) \right] \frac{\varepsilon}{f(b) - f(a)} = \varepsilon$$

Observation: if $f : [a,b] \to \mathbb{R}$ cont.

it is actually uniformly continuous

i.e. $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$

whenever $|x - y| < \delta$

Theorem $f : [a,b] \to \mathbb{R}$ continuous $\Rightarrow$ f integrable

Proof pick $\varepsilon > 0$

unif. conti $\Rightarrow \exists \delta$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$

whenever $|x - y| < \delta$

pick n s.t. $\frac{b-a}{n} < \delta$

Define partition $P = \{t_k\}$ by

$$t_k = a + k \frac{b-a}{n}$$

$[t_{k-1}, t_k]$ is closed

$\Rightarrow \exists\ x_1$ and x_2 s.t. $\quad f(x_1) = M(f, [t_{k-1}, t_k])$

$f(x_2) = m(f, [t_{k-1}, t_k])$

as $|x_1 - x_2| \leq t_k - t_{k-1} = \frac{b-a}{n} < \delta$

$$\Rightarrow \boxed{M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) = f(x_1) - f(x_2) < \frac{\varepsilon}{b-a}}$$

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \underbrace{\left[M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\right]}_{< \frac{\varepsilon}{b-a}} \underbrace{(t_k - t_{k-1})}_{= \frac{b-a}{n}}$$

$$< \sum_{k=1}^{n} \frac{\varepsilon}{\cancel{b-a}} \cdot \frac{\cancel{b-a}}{n} = \varepsilon$$

Properties of integrable functions

want to show: f integrable $\Rightarrow$ cf integrable for any constant c

Lemma I closed interval, $f: I \to \mathbb{R}$ bounded, $c \in \mathbb{R}$, $c \neq 0$

$$M(cf, I) - m(cf, I) = |c| \, \big(M(f,I) - m(f,I) \big)$$

proof. If $c>0$:

$$\begin{aligned} M(cf,I) &= \sup\{cf(x), x\in I\} \\ &= c \sup\{f(x), x \in I\} \\ &= c\, M(f,I) \end{aligned}$$

$$m(cf,I) = \dots = c\, m(f,I)$$

If $c<0$:

$$\begin{aligned} M(cf,I) &= \sup\{c f(x), x\in I\} \\ &= c \inf\{f(x), x\in I\} = c\, m(f,I) \end{aligned}$$

$c<0$!

(c reverses $\leq$ sign)

$$m(cf,I) = \dots = c\, M(f,I)$$

check statement for $c<0$:

$$\begin{aligned} M(cf,I) - m(cf,I) &= c\, m(f,I) - c\, M(f,I) \\ &= |c| \big(M(f,I) - m(f,I) \big) \quad \text{as } c<0 \end{aligned}$$

Theorem: Let f, g be integrable functions on $[a,b]$, let $c \in \mathbb{R}$

(a) cf is integrable

(b) $f+g$ is integrable

Proof (a) if $c=0$ statement trivially true.

$c \neq 0$ pick $\varepsilon > 0$

f integrable $\Rightarrow \exists$ partition P s.t.

$$U(f,P) - L(f,P) < \frac{\varepsilon}{|c|}$$

$$\Rightarrow U(cf,P) - L(cf,P) \underset{\text{Lemma}}{=} |c| \left(U(f,P) - L(f,P) \right)$$

$$< |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon \checkmark$$

(b) Recall: $\overset{I \text{ interval}}{}$ $M(f+g, I) = \sup\{f(x)+g(x),\ x\in I\}$

$$\leq \sup\{f(x),\ x\in I\} + \sup\{g(x),\ x\in I\}$$

$$= M(f, I) + M(g, I)$$

similarly $m(f+g, I) \geq m(f, I) + m(g, I)$

$$\Rightarrow \quad U(f+g, P) = \sum_k M(f+g, [t_{k-1}, t_k])$$

$$\leq \sum_k M(f, [t_{k-1}, t_k]) + M(g, [t_{k-1}, t_k])$$

$$= U(f, P) + U(g, P)$$

Similarly

$$L(f+g, P) \geq L(f, P) + L(g, P)$$